Name:	Date:	Marks:

3.51 Integration by Substitution

Learning Objectives

Students should be able to:

Checklist Core material

☐ Extend the idea of 'reverse differentiation' to include the integration of e^{ax+b}, $\frac{1}{ax+b}$, sin(ax+b), cos(ax+b) and $\sec^2(ax+b)$ and $\frac{1}{x^2+a^2}$.
e.g. Including examples such as $\frac{1}{3x^2+2}$

☐ Use trigonometrical relationships in carrying out integration
e.g. use of double–angle formulae to integrate $\sin^2 x$ or $\cos^2(2x)$

☐ recognise an integrand of the form $\frac{kf'(x)}{f(x)}$, and such functions
e.g. integration of $\frac{x}{x^2+1}$ or tan x

☐ Use a given substitution to simplify and evaluate either a definite or an indefinite integral
e.g. to integrate $\sin^2 2x \cos x$ using the substitution $u = \sin x$

Nos	Questions	References

1 **(i)** By first expanding $\sin(2x+x)$, show that $\sin 3x \equiv 3\sin x - 4\sin^3 x$. [4]

(ii) Hence, showing all necessary working, find the exact value of $\int_0^{\frac{1}{3}\pi} \sin^3 x \, dx$. [4]

2 Let $I = \int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{x}{1-x}\right)} \, dx$.

(i) Using the substitution $x = \cos^2\theta$, show that $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\cos^2\theta \, d\theta$. [4]

(ii) Hence find the exact value of I. [4]

3 **(i)** Show that $\frac{2\sin x - \sin 2x}{1-\cos 2x} \equiv \frac{\sin x}{1+\cos x}$. [4]

(ii) Hence, showing all necessary working, find $\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{2\sin x - \sin 2x}{1-\cos 2x} \, dx$, giving your answer in the form $\ln k$. [4]

4 **(i)** Using the expansions of $\cos(3x+x)$ and $\cos(3x-x)$, show that

$$\tfrac{1}{2}(\cos 4x + \cos 2x) \equiv \cos 3x \cos x.$$ [3]

(ii) Hence show that $\int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} \cos 3x \cos x \, dx = \frac{3}{8}\sqrt{3}$. [3]

5 It is given that $x = \ln(1-y) - \ln y$, where $0 < y < 1$.

(i) Show that $y = \frac{e^{-x}}{1+e^{-x}}$. [2]

(ii) Hence show that $\int_0^1 y \, dx = \ln\left(\frac{2e}{e+1}\right)$. [4]

6 **(i)** Prove that if $y = \frac{1}{\cos\theta}$ then $\frac{dy}{d\theta} = \sec\theta\tan\theta$. [2]

(ii) Prove the identity $\frac{1+\sin\theta}{1-\sin\theta} \equiv 2\sec^2\theta + 2\sec\theta\tan\theta - 1$. [3]

(iii) Hence find the exact value of $\int_0^{\frac{1}{4}\pi} \frac{1+\sin\theta}{1-\sin\theta} \, d\theta$. [4]

Nos	Questions	References
7	**(i)** Prove the identity $\tan 2\theta - \tan\theta \equiv \tan\theta \sec 2\theta$.	[4]
	(ii) Hence show that $\int_0^{\frac{1}{6}\pi} \tan\theta \sec 2\theta \, d\theta = \frac{1}{2}\ln\frac{3}{2}$.	[4]
8	Let $I = \int_1^4 \frac{(\sqrt{x}) - 1}{2(x + \sqrt{x})} \, dx$.	
	(i) Using the substitution $u = \sqrt{x}$, show that $I = \int_1^2 \frac{u-1}{u+1} \, du$.	[3]
	(ii) Hence show that $I = 1 + \ln\frac{4}{9}$.	[6]
9	Let $I = \int_0^1 \frac{x^5}{(1+x^2)^3} \, dx$.	
	(i) Using the substitution $u = 1 + x^2$, show that $I = \int_1^2 \frac{(u-1)^2}{2u^3} \, du$.	[3]
	(ii) Hence find the exact value of I.	[5]
10	Let $I = \int_0^1 \frac{9}{(3+x^2)^2} \, dx$.	
	(i) Using the substitution $x = (\sqrt{3})\tan\theta$, show that $I = \sqrt{3}\int_0^{\frac{1}{6}\pi} \cos^2\theta \, d\theta$.	[3]
	(ii) Hence find the exact value of I.	[4]
11	Use the substitution $u = 4 - 3\cos x$ to find the exact value of $\int_0^{\frac{1}{2}\pi} \frac{9\sin 2x}{\sqrt{(4 - 3\cos x)}} \, dx$.	[8]
12	**(a)** Find $\int (4 + \tan^2 2x) \, dx$.	[3]
	(b) Find the exact value of $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin(x + \frac{1}{6}\pi)}{\sin x} \, dx$.	[5]

Nos	Questions	References

13 Let $I = \int_0^1 \frac{\sqrt{x}}{2-\sqrt{x}}\,\mathrm{d}x$.

(i) Using the substitution $u = 2 - \sqrt{x}$, show that $I = \int_1^2 \frac{2(2-u)^2}{u}\,\mathrm{d}u$. [4]

(ii) Hence show that $I = 8\ln 2 - 5$. [4]

14 By first using the substitution $u = \mathrm{e}^x$, show that

$$\int_0^{\ln 4} \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x} + 3\mathrm{e}^x + 2}\,\mathrm{d}x = \ln\left(\tfrac{8}{5}\right).$$ [10]

15 Use the substitution $u = 1 + 3\tan x$ to find the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1+3\tan x)}}{\cos^2 x}\,\mathrm{d}x.$$ [5]

16 **(i)** Prove that $\cot\theta + \tan\theta \equiv 2\operatorname{cosec} 2\theta$. [3]

(ii) Hence show that $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \operatorname{cosec} 2\theta\,\mathrm{d}\theta = \frac{1}{2}\ln 3$. [4]

17 Use the substitution $u = 3x + 1$ to find $\int \frac{3x}{3x+1}\,\mathrm{d}x$. [4]

18 Use the substitution $u = \sin 4x$ to find the exact value of $\int_0^{\frac{1}{24}\pi} \cos^3 4x\,\mathrm{d}x$. [5]

19 Express $4\cos\theta + 3\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

Hence find $\int \frac{50}{(4\cos\theta + 3\sin\theta)^2}\,\mathrm{d}\theta$. [3]

20 **(i)** By differentiating $\frac{1}{\cos x}$, show that the derivative of $\sec x$ is $\sec x \tan x$. Hence show that if $y = \ln(\sec x + \tan x)$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec x$. [4]

(ii) Using the substitution $x = (\sqrt{3})\tan\theta$, find the exact value of

$$\int_1^3 \frac{1}{\sqrt{(3+x^2)}}\,\mathrm{d}x,$$

expressing your answer as a single logarithm. [4]

Nos	Questions	References

21 **(i)** Express $(\sqrt{3})\cos x + \sin x$ in the form $R\cos(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \frac{1}{((\sqrt{3})\cos x + \sin x)^2}\,dx = \tfrac{1}{4}\sqrt{3}.$$ [4]

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(i) By differentiating $\frac{1}{\cos x}$, show that if $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$. [2]

(ii) Show that $\frac{1}{\sec x - \tan x} \equiv \sec x + \tan x$. [1]

(iii) Deduce that $\frac{1}{(\sec x - \tan x)^2} \equiv 2\sec^2 x - 1 + 2\sec x \tan x$. [2]

(iv) Hence show that $\int_0^{\frac{1}{4}\pi} \frac{1}{(\sec x - \tan x)^2}\,dx = \frac{1}{4}(8\sqrt{2} - \pi)$. [3]

23 **(i)** Use the substitution $u = \tan x$ to show that, for $n \neq -1$,

$$\int_0^{\frac{1}{4}\pi} (\tan^{n+2} x + \tan^n x)\,dx = \frac{1}{n+1}.$$ [4]

(ii) Hence find the exact value of

(a) $\int_0^{\frac{1}{4}\pi} (\sec^4 x - \sec^2 x)\,dx$, [3]

(b) $\int_0^{\frac{1}{4}\pi} (\tan^9 x + 5\tan^7 x + 5\tan^5 x + \tan^3 x)\,dx$. [3]

24 **(i)** Prove the identity $\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3$. [4]

(ii) Hence

(a) solve the equation $\cos 4\theta + 4\cos 2\theta = 1$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, [3]

(b) find the exact value of $\int_0^{\frac{1}{4}\pi} \cos^4\theta\,d\theta$. [3]

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Let $I = \int_0^1 \frac{x^2}{\sqrt{(4-x^2)}}\,dx$.

(i) Using the substitution $x = 2\sin\theta$, show that

$$I = \int_0^{\frac{1}{6}\pi} 4\sin^2\theta\,d\theta.$$ [3]

(ii) Hence find the exact value of I. [4]

Nos	Questions	References
26	It is given that $f(x) = 4\cos^2 3x$.	
	(i) Find the exact value of $f'\left(\frac{1}{9}\pi\right)$. [3]	
	(ii) Find $\int f(x)\,dx$. [3]	
27	**(i)** Using the expansions of $\cos(3x - x)$ and $\cos(3x + x)$, prove that $$\tfrac{1}{2}(\cos 2x - \cos 4x) \equiv \sin 3x \sin x.$$ [3]	
	(ii) Hence show that $$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin 3x \sin x\,dx = \tfrac{1}{8}\sqrt{3}.$$ [3]	
28	**(i)** Prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [4]	
	(ii) Using this result, find the exact value of $$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \cos^3\theta\,d\theta.$$ [4]	
29	**(i)** Prove the identity $\cos 4\theta - 4\cos 2\theta + 3 \equiv 8\sin^4\theta$. [4]	
	(ii) Using this result find, in simplified form, the exact value of $$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin^4\theta\,d\theta.$$ [4]	
30	**(i)** Use the substitution $x = 2\tan\theta$ to show that $$\int_0^2 \frac{8}{(4+x^2)^2}\,dx = \int_0^{\frac{1}{4}\pi} \cos^2\theta\,d\theta.$$ [4]	
	(ii) Hence find the exact value of $$\int_0^2 \frac{8}{(4+x^2)^2}\,dx.$$ [4]	
31	Find the exact value of the constant k for which $\int_1^k \frac{1}{2x-1}\,dx = 1$. [4]	

Nos	Questions	References
32	**(i)** Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3] **(ii)** Hence show that $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{1}{\left(\cos\theta + (\sqrt{3})\sin\theta\right)^2}\,d\theta = \frac{1}{\sqrt{3}}$. [4]	
33	**(i)** Use the substitution $x = \sin^2\theta$ to show that $$\int \sqrt{\left(\frac{x}{1-x}\right)}\,dx = \int 2\sin^2\theta\,d\theta.$$ [4] **(ii)** Hence find the exact value of $$\int_0^{\frac{1}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,dx.$$ [4]	
34	**(i)** Use the substitution $x = \tan\theta$ to show that $$\int \frac{1-x^2}{(1+x^2)^2}\,dx = \int \cos 2\theta\,d\theta.$$ [4] **(ii)** Hence find the value of $$\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx.$$ [3]	
35	**(i)** Prove the identity $$\sin^2\theta\cos^2\theta \equiv \tfrac{1}{8}(1-\cos 4\theta).$$ [3] **(ii)** Hence find the exact value of $$\int_0^{\frac{1}{3}\pi} \sin^2\theta\cos^2\theta\,d\theta.$$ [3]	
36	**(i)** Prove the identity $$\cot x - \cot 2x \equiv \operatorname{cosec} 2x.$$ [3] **(ii)** Show that $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot x\,dx = \frac{1}{2}\ln 2$. [3] **(iii)** Find the exact value of $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \operatorname{cosec} 2x\,dx$, giving your answer in the form $a\ln b$. [4]	

Nos	Answers	Nos	Answers
1 (ii)	$\frac{5}{24}$	24 (ii) (a)	0.572, -0.572
		(b)	$\frac{3\pi}{32}+\frac{1}{4}$
2 (ii)	$\frac{\pi}{6}$	25 (ii)	$\frac{\pi}{3}-\frac{\sqrt{3}}{2}$
3 (ii)	$\ln\left(\frac{3}{2}\right)$	26 (i)	$-6\sqrt{3}$
		(ii)	$2x+\frac{1}{3}\sin 6x+c$
6 (iii)	$2\sqrt{2}-\frac{\pi}{6}$	28 (ii)	$\frac{2}{3}-\frac{3\sqrt{3}}{8}$
9 (ii)	$\frac{1}{2}\ln 2-\frac{5}{16}$	29 (ii)	$\frac{\pi}{16}-\frac{\sqrt{3}}{32}$
10 (ii)	$\frac{\sqrt{3}}{12}\pi+\frac{3}{8}$	30 (ii)	$\frac{\pi}{8}+\frac{1}{4}$
11	$\frac{20}{3}$	31	$\frac{1}{2}(e^2+1)$
12 (a)	$3x+\frac{1}{2}\tan 2x+c$	32 (i)	$R=2$, $\alpha=\frac{\pi}{6}$
(b)	$\frac{\sqrt{3}}{8}\pi-\frac{1}{2}\ln\frac{1}{\sqrt{2}}$	33 (ii)	$\frac{\pi}{6}-\frac{\sqrt{3}}{4}$
15	$\frac{14}{9}$	34 (ii)	$\frac{1}{2}$
17	$\frac{1}{3}(3x+1)-\frac{1}{3}\ln(3x+1)+c$	35 (ii)	$\frac{1}{8}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right)$
18	$\frac{11}{96}$	36 (iii)	$\frac{1}{4}\ln 3$
19	$2\tan(\theta-0.6435)$		
20 (ii)	$\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$		
21 (i)	$R=2$, $\alpha=\frac{\pi}{6}$		
23 (ii) (a)	$\frac{1}{3}$		
(b)	$\frac{25}{24}$		