| Name: | Date: | Marks: |
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# 3.41 Differentiation I

## Learning Objectives

### Students should be able to:

| Checklist | Students should be able to: |
|---|---|
| ☐ | Use the derivatives of $e^x$, ln x, sin x, cos x, tan x, together with constant multiples, sums, differences and composites. Derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ are not required. |
| ☐ | Differentiate products and quotients. e.g. $\frac{2x-4}{3x+2}$, $x^2 \ln x$, $xe^{1-x^2}$. Including use in problems involving tangents and normals. |

| Nos | Questions | Reference |
|---|---|---|
| 1 | The equation of a curve is $2x^2y - xy^2 = a^3$, where $a$ is a positive constant. Show that there is only one point on the curve at which the tangent is parallel to the $x$-axis and find the $y$-coordinate of this point. [7]<br>Ans: $y = -2a$ | |
| 2 | Find the gradient of the curve $x^3 + 3xy^2 - y^3 = 1$ at the point with coordinates (1, 3). [4]<br>Ans: $\frac{10}{3}$ | |
| 3 | Find the exact coordinates of the point on the curve $y = \frac{x}{1 + \ln x}$ at which the gradient of the tangent is equal to $\frac{1}{4}$. [7]<br>Ans: $(e, \frac{1}{2}e)$ | |
| 4 | The equation of a curve is $y = \frac{1 + e^{-x}}{1 - e^{-x}}$, for $x > 0$.<br>**(i)** Show that $\frac{dy}{dx}$ is always negative. [3]<br>**(ii)** The gradient of the curve is equal to $-1$ when $x = a$. Show that $a$ satisfies the equation $e^{2a} - 4e^a + 1 = 0$. Hence find the exact value of $a$. [4]<br>Ans: $a = \ln(2 + \sqrt{3})$ | |
| 5 | The variables $x$ and $y$ satisfy the relation $\sin y = \tan x$, where $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$. Show that $$\frac{dy}{dx} = \frac{1}{\cos x \sqrt{(\cos 2x)}}.$$ [5] | |
| 6 | A curve has equation $y = \frac{e^{3x}}{\tan \frac{1}{2}x}$. Find the $x$-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. Give your answers correct to 3 decimal places. [6]<br>Ans: $x = 0.340$, $x = 2.802$ | |
| 7 | The equation of a curve is $x^2(x + 3y) - y^3 = 3$.<br>**(i)** Show that $\frac{dy}{dx} = \frac{x^2 + 2xy}{y^2 - x^2}$. [4]<br>**(ii)** Hence find the exact coordinates of the two points on the curve at which the gradient of the normal is 1. [4]<br>Ans: $(\sqrt[3]{3}, 0)$ and $(1, -2)$ | |
| 8 | The equation of a curve is $2x^4 + xy^3 + y^4 = 10$.<br>**(i)** Show that $\frac{dy}{dx} = -\frac{8x^3 + y^3}{3xy^2 + 4y^3}$. [4]<br>**(ii)** Hence show that there are two points on the curve at which the tangent is parallel to the $x$-axis and find the coordinates of these points. [4]<br>Ans: $(-1, 2)$ and $(1, -2)$ | |

| Nos | Questions | Reference |
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9 The curve with equation $y = \dfrac{2 - \sin x}{\cos x}$ has one stationary point in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

**(i)** Find the exact coordinates of this point. [5]

Ans: $(\frac{\pi}{6}, \sqrt{3})$

**(ii)** Determine whether this point is a maximum or a minimum point. [2]

Ans: point is a minimum point

10 The equation of a curve is $x^3y - 3xy^3 = 2a^4$, where $a$ is a non-zero constant.

**(i)** Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - 3y^3}{9xy^2 - x^3}$. [4]

**(ii)** Hence show that there are only two points on the curve at which the tangent is parallel to the $x$-axis and find the coordinates of these points. [4]

Ans: $(-a, a)$ and $(a, -a)$

11 A curve has equation $y = \frac{2}{3}\ln(1 + 3\cos^2 x)$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

**(i)** Express $\dfrac{dy}{dx}$ in terms of $\tan x$. [4]

Ans: $\dfrac{dy}{dx} = -\dfrac{4\tan x}{4 + \tan^2 x}$

**(ii)** Hence find the $x$-coordinate of the point on the curve where the gradient is $-1$. Give your answer correct to 3 significant figures. [2]

Ans: $x = 1.11$

12 The curve with equation $y = e^{-ax}\tan x$, where $a$ is a positive constant, has only one point in the interval $0 < x < \frac{1}{2}\pi$ at which the tangent is parallel to the $x$-axis. Find the value of $a$ and state the exact value of the $x$-coordinate of this point. [7]

Ans: $a = 2$ and $x = \dfrac{\pi}{4}$

13 The equation of a curve is $xy(x - 6y) = 9a^3$, where $a$ is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the $x$-axis, and find the coordinates of this point. [7]

Ans: $(-3a, -a)$

14 The equation of a curve is $y = \dfrac{\sin x}{1 + \cos x}$, for $-\pi < x < \pi$. Show that the gradient of the curve is positive for all $x$ in the given interval. [4]

Ans: $1 + \cos x$ is always positive, $\dfrac{1}{1 + \cos x}$ is always positive

15 The curve with equation $y = \sin x \cos 2x$ has one stationary point in the interval $0 < x < \frac{1}{2}\pi$. Find the $x$-coordinate of this point, giving your answer correct to 3 significant figures. [6]

Ans: $x = 0.421$

| Nos | Questions | Reference |
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16 The equation of a curve is $x^3 - 3x^2y + y^3 = 3$.

**(i)** Show that $\frac{dy}{dx} = \frac{x^2 - 2xy}{x^2 - y^2}$. [4]

**(ii)** Find the coordinates of the points on the curve where the tangent is parallel to the $x$-axis. [5]

Ans: $(-2, -1)$ and $(0, \sqrt[3]{3})$

17 The curve with equation $y = \frac{(\ln x)^2}{x}$ has two stationary points. Find the exact values of the coordinates of these points. [6]

Ans: $(1, 0)$ and $(e^2, 4e^{-2})$

18 A curve has equation

$$\sin y \ln x = x - 2\sin y,$$

for $-\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi$.

**(i)** Find $\frac{dy}{dx}$ in terms of $x$ and $y$. [5]

Ans: $\frac{dy}{dx} = \frac{x - \sin y}{4(\ln x \cos y + 2\cos y)}$

**(ii)** Hence find the exact $x$-coordinate of the point on the curve at which the tangent is parallel to the $x$-axis. [3]

Ans: $x = \frac{1}{e}$

19 The equation of a curve is $y = e^{-2x}\tan x$, for $0 \leqslant x < \frac{1}{2}\pi$.

**(i)** Obtain an expression for $\frac{dy}{dx}$ and show that it can be written in the form $e^{-2x}(a + b\tan x)^2$, where $a$ and $b$ are constants. [5]

Ans: $e^{-2x}(\tan x - 1)^2$

**(ii)** Explain why the gradient of the curve is never negative. [1]

Ans: $e^{-2x}$ is always positive , $(\tan x - 1)^2$ is always positive

**(iii)** Find the value of $x$ for which the gradient is least. [1]

Ans: $x = \frac{\pi}{4}$

20 The equation of a curve is

$$y = 3\cos 2x + 7\sin x + 2.$$

Find the $x$-coordinates of the stationary points in the interval $0 \leqslant x \leqslant \pi$. Give each answer correct to 3 significant figures. [7]

Ans: $x = 0.623, 2.52, 1.57$

21 A curve has equation $y = \cos x \cos 2x$. Find the $x$-coordinate of the stationary point on the curve in the interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 significant figures. [6]

Ans: $x = 1.15$

22 The curve with equation $y = \frac{e^{2x}}{4 + e^{3x}}$ has one stationary point. Find the exact values of the coordinates of this point. [6]

Ans: $(\ln 2, \frac{1}{3})$

| Nos | Questions | Reference |
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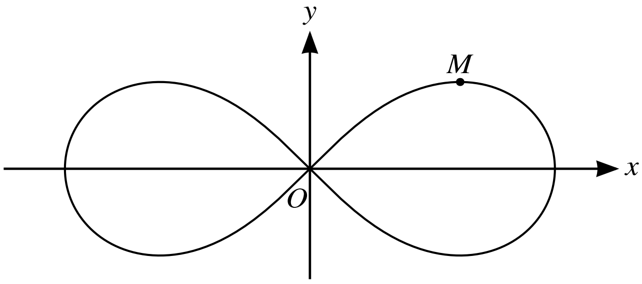
23




The diagram shows the curve $(x^2 + y^2)^2 = 2(x^2 - y^2)$ and one of its maximum points $M$. Find the coordinates of $M$. [7]

Ans: $(\frac{\sqrt{3}}{2}, \frac{1}{2})$

24 The equation of a curve is $y = \frac{1+x}{1+2x}$ for $x > -\frac{1}{2}$. Show that the gradient of the curve is always negative. [3]

Ans: $\frac{dy}{dx} = -\frac{1}{(1+2x)^2}$ is always negative

25 A curve has equation $3e^{2x}y + e^{x}y^3 = 14$. Find the gradient of the curve at the point (0, 2). [5]

Ans: $\frac{dy}{dx} = -\frac{4}{3}$

26 For each of the following curves, find the gradient at the point where the curve crosses the $y$-axis:

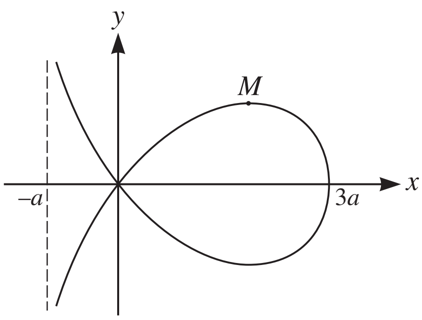
**(i)** $y = \frac{1+x^2}{1+e^{2x}}$; [3]

Ans: $\frac{dy}{dx} = -\frac{1}{2}$

**(ii)** $2x^3 + 5xy + y^3 = 8$. [4]

Ans: $\frac{dy}{dx} = -\frac{5}{6}$

27




The diagram shows the curve with equation

$$x^3 + xy^2 + ay^2 - 3ax^2 = 0,$$

where $a$ is a positive constant. The maximum point on the curve is $M$. Find the $x$-coordinate of $M$ in terms of $a$. [6]

Ans: $x = \sqrt{3}a$

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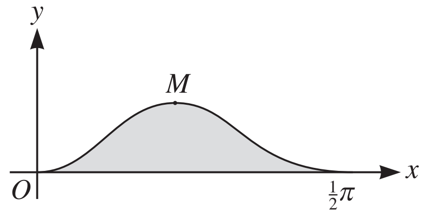
28




The diagram shows the curve $y = \sin^2 2x \cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point $M$.

Find the $x$-coordinate of $M$. [6]

Ans: $x = 0.685$

29 The equation of a curve is $\ln(xy) - y^3 = 1$.

**(i)** Show that $\frac{dy}{dx} = \frac{y}{x(3y^3 - 1)}$. [4]

**(ii)** Find the coordinates of the point where the tangent to the curve is parallel to the $y$-axis, giving each coordinate correct to 3 significant figures. [4]

Ans: (5.47, 0.693)

30 The equation of a curve is $3x^2 - 4xy + y^2 = 45$.

**(i)** Find the gradient of the curve at the point $(2, -3)$. [4]

Ans: $\frac{dy}{dx} = \frac{4y - 6x}{2y - 4x} = \frac{12}{7}$

**(ii)** Show that there are no points on the curve at which the gradient is 1. [3]

Ans: sub $y = x$ into $3x^2 - 4xy + y^2 = 45$

31 The equation of a curve is $y = 3\sin x + 4\cos^3 x$.

**(i)** Find the $x$-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. [6]

Ans: $x = \frac{1}{12}\pi, \frac{5}{12}\pi, \frac{1}{2}\pi$

**(ii)** Determine the nature of the stationary point in this interval for which $x$ is least. [2]

Ans: maximum point at $x = \frac{1}{12}\pi$

32 The curve with equation $y = \frac{e^{2x}}{x^3}$ has one stationary point.

**(i)** Find the $x$-coordinate of this point. [4]

Ans: $x = \frac{3}{2}$

**(ii)** Determine whether this point is a maximum or a minimum point. [2]

Ans: minimum point at $x = \frac{3}{2}$

33 The equation of a curve is $y = \frac{e^{2x}}{1 + e^{2x}}$. Show that the gradient of the curve at the point for which $x = \ln 3$ is $\frac{9}{50}$. [4]

| Nos | Questions | Reference |
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34 Find $\frac{dy}{dx}$ in each of the following cases:

**(i)** $y = \ln(1 + \sin 2x)$, [2]

Ans: $\frac{dy}{dx} = \frac{2\cos 2x}{1+\sin 2x}$

**(ii)** $y = \frac{\tan x}{x}$. [2]

Ans: $\frac{dy}{dx} = \frac{x\sec^2 x - \tan x}{x^2}$

35 The curve with equation

$$6e^{2x} + ke^{y} + e^{2y} = c,$$

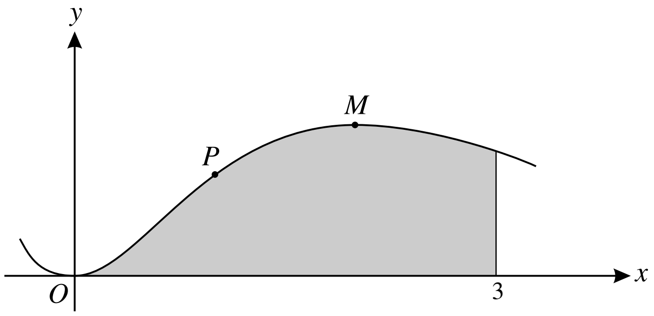
where $k$ and $c$ are constants, passes through the point $P$ with coordinates $(\ln 3, \ln 2)$.

**(i)** Show that $58 + 2k = c$. [2]

**(ii)** Given also that the gradient of the curve at $P$ is $-6$, find the values of $k$ and $c$. [5]

Ans: $k = 5$, $c = 68$

36




The diagram shows the curve $y = x^2 e^{-x}$.

**(i)** Find the $x$-coordinate of the maximum point $M$ on the curve. [4]

Ans: $x = 2$

**(ii)** Find the $x$-coordinate of the point $P$ at which the tangent to the curve passes through the origin. [2]

Ans: $x = 1$

37 The curve $y = \frac{\ln x}{x^3}$ has one stationary point. Find the $x$-coordinate of this point. [4]

Ans: $x = e^{\frac{1}{3}}$

| Nos | Questions | Reference |
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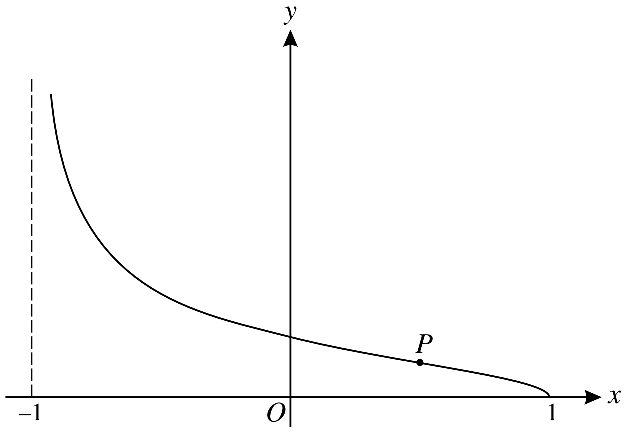
38




The diagram shows the curve $y = \sqrt{\left(\frac{1-x}{1+x}\right)}$.

**(i)** By first differentiating $\frac{1-x}{1+x}$, obtain an expression for $\frac{dy}{dx}$ in terms of $x$. Hence show that the gradient of the normal to the curve at the point $(x, y)$ is $(1+x)\sqrt{(1-x^2)}$. [5]

**(ii)** The gradient of the normal to the curve has its maximum value at the point $P$ shown in the diagram. Find, by differentiation, the $x$-coordinate of $P$. [4]

Ans: $x = \frac{1}{2}$

39 The equation of a curve is

$$x \ln y = 2x + 1.$$

**(i)** Show that $\frac{dy}{dx} = -\frac{y}{x^2}$. [4]

**(ii)** Find the equation of the tangent to the curve at the point where $y = 1$, giving your answer in the form $ax + by + c = 0$. [4]

Ans: $y + 4x + 1 = 0$

40 The variables $x$ and $y$ satisfy the equation $y^3 = Ae^{2x}$, where $A$ is a constant. The graph of $\ln y$ against $x$ is a straight line.

**(i)** Find the gradient of this line. [2]

Ans: $\frac{dy}{dx} = \frac{2}{3}$

**(ii)** Given that the line intersects the axis of $\ln y$ at the point where $\ln y = 0.5$, find the value of $A$ correct to 2 decimal places. [2]

Ans: $A = 4.48$

| Nos | Questions | Reference |
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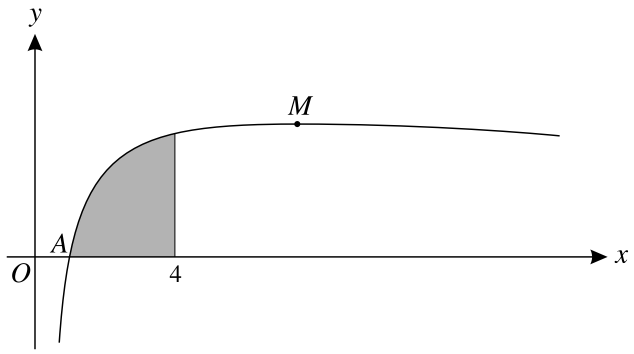
41




The diagram shows the curve $y = \frac{\ln x}{\sqrt{x}}$ and its maximum point $M$. The curve cuts the $x$-axis at the point $A$.

**(i)** State the coordinates of $A$. [1]

Ans: (1, 0)

**(ii)** Find the exact value of the $x$-coordinate of $M$. [4]

Ans: $x = e^2$

42 The equation of a curve is $x^3 - x^2y - y^3 = 3$.

**(i)** Find $\frac{dy}{dx}$ in terms of $x$ and $y$. [4]

Ans: $\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 3y^2}$

**(ii)** Find the equation of the tangent to the curve at the point (2, 1), giving your answer in the form $ax + by + c = 0$. [2]

Ans: $7y - 8x + 9 = 0$

43 The equation of a curve is $xy(x + y) = 2a^3$, where $a$ is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the $x$-axis, and find the coordinates of this point. [8]

Ans: $(a, -2a)$

44 The curve with equation $y = e^{-x}\sin x$ has one stationary point for which $0 \leqslant x \leqslant \pi$.

**(i)** Find the $x$-coordinate of this point. [4]

Ans: $x = \frac{\pi}{4}$

**(ii)** Determine whether this point is a maximum or a minimum point. [2]

Ans: maximum point at $x = \frac{1}{4}\pi$

45 The equation of a curve is $y = x\sin 2x$, where $x$ is in radians. Find the equation of the tangent to the curve at the point where $x = \frac{1}{4}\pi$. [4]

Ans: $y = x$

| Nos | Questions | Reference |
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46 The curve with equation $y = 6e^x - e^{3x}$ has one stationary point.

**(i)** Find the $x$-coordinate of this point. [4]

Ans: $x = \frac{1}{2}\ln 2$

**(ii)** Determine whether this point is a maximum or a minimum point. [2]

Ans: maximum point at $x = \frac{1}{2}\ln 2$

47 The equation of a curve is $x^3 + 2y^3 = 3xy$.

**(i)** Show that $\frac{dy}{dx} = \frac{y - x^2}{2y^2 - x}$. [4]

**(ii)** Find the coordinates of the point, other than the origin, where the curve has a tangent which is parallel to the $x$-axis. [5]

Ans: (1, 1)

48 The equation of a curve is $y = x + \cos 2x$. Find the $x$-coordinates of the stationary points of the curve for which $0 \leqslant x \leqslant \pi$, and determine the nature of each of these stationary points. [7]

Ans: $x = \frac{\pi}{12}$ is a maximum point , $x = \frac{5\pi}{12}$ is a minimum point

49 Find the gradient of the curve with equation

$$2x^2 - 4xy + 3y^2 = 3,$$

at the point (2, 1). [4]

Ans: $\frac{dy}{dx} = 2$

50 The equation of a curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a},$$

where $a$ is a positive constant.

**(i)** Express $\frac{dy}{dx}$ in terms of $x$ and $y$. [3]

Ans: $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

**(ii)** The straight line with equation $y = x$ intersects the curve at the point $P$. Find the equation of the tangent to the curve at $P$. [3]

Ans: $x + y = \frac{1}{2}a$

51 The curve $y = e^x + 4e^{-2x}$ has one stationary point.

**(i)** Find the $x$-coordinate of this point. [4]

Ans: $x = \ln 2$

**(ii)** Determine whether the stationary point is a maximum or a minimum point. [2]

Ans: minimum point at $x = \ln 2$

52 The equation of a curve is $y = 2\cos x + \sin 2x$. Find the $x$-coordinates of the stationary points on the curve for which $0 < x < \pi$, and determine the nature of each of these stationary points. [7]

Ans: maximum point at $x = \frac{\pi}{6}$ , minimum point at $x = \frac{5\pi}{6}$